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Engineering Notes

Solution of the Flutter Eigenvalue Problem with Mixed Structural/ Aerodynamic Uncertainty

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Nomenclature

c_p	=	pressure coefficient
D	=	uncertainty set
F	=	flutter matrix
g	=	nondimensional damping
J	=	mass moment of inertia
K	=	stiffness matrix
k	=	reduced frequency
L	=	reference length
M	=	mass matrix
m	=	mass
P	=	system matrix
p	=	eigenvalue
Q	=	aerodynamic matrix
V	=	airspeed
w	=	uncertainty bound
x	=	center of mass
Δ	=	uncertainty matrix
δ	=	uncertainty parameter
η	=	modal coordinates
μ	=	structured singular value
ρ	=	air density

I. Introduction

A RECENT development in the field of aeroelastic analysis considering model uncertainty is the so-called μ-p method [1]. The basic principle of the μ-p method is to exploit structured-singular-value (or μ) analysis [2] to investigate if the system uncertainties can make the flutter determinant zero for a given flutter eigenvalue p. This makes it possible to compute sets of feasible eigenvalues in the complex plane, that can be used to predict damping bounds and perform robust flutter analysis. In contrast to previous formulations based on μ analysis [3–5], the μ-p method thus extends standard linear flutter analysis [6,7] to take deterministic uncertainty and variation into account.

In the studies published so far [1,8–10] only complex valued aerodynamic perturbations have been included in the analysis, which is known to be a favorable case for μ analysis [11]. For this reason, the primary objective of this study is to investigate what impact mixed structural/aerodynamic (real/complex) perturbations may have on the quality of the robust flutter solutions. A generic delta-wing wind-tunnel model with an external wing-tip store is used for this purpose, where uncertainties in the aerodynamic and mass properties are taken into account.

In practice, the structured singular value μ has to be estimated by a computable upper bound, which can lead to some amount of conservativeness in the analysis (in particular for mixed real/complex uncertainty). In this study, the impact of this approximation is investigated by comparing exact p-k eigenvalue sets from parameter variation with eigenvalue sets obtained from μ-p analysis. It is found that the μ-p eigenvalue sets closely bound the exact eigenvalue sets in the complex plane, verifying the accuracy and practical usefulness of μ-p analysis.

II. Test Case

A delta-wing wind-tunnel model, as shown in Fig. 1, is used as a test case. As described in more detail in [10], the model has a simple structural design and is made of glass-fiber and carbon-fiber composite materials. It has a semispan of 0.88 m and mean chord of 0.70 m and is mounted vertically in the low-speed wind tunnel L2000 at the Royal Institute of Technology. To allow for investigations of external store dynamics, the wing is equipped with a pylon and a missile at the wing tip.

The numerical analysis is based on a NASTRAN [12] model with shell elements for the wing, mass elements for the missile, and aerodynamic panels for doublet-lattice aerodynamic loads. The relatively stiff missile was modeled as a rigid body attached to the wing tip. Twenty structural modes were used to establish a standard modal formulation of the flutter problem. To allow for subsequent μ-p analysis in MATLAB [13], the required matrices and other data were extracted from NASTRAN and stored in a format that could then be loaded into the MATLAB workspace.

The present configuration of the wind-tunnel model suffers a 5.0 Hz flutter instability in the second aeroelastic mode at about 32.5–33.0 m/s. Because of the simple structural design, a nominal p-k flutter analysis (without uncertainty) predicts the fairly accurate values of 5.2 Hz and 32.0 m/s, respectively. By taking aerodynamic uncertainty into account and performing successive model validation in the test, the robust flutter speed predicted by μ-p analysis differs by less than 1 m/s from the experimental value [10]. The present study will also take uncertainty in the missile's mass properties into account, with the objective to assess the accuracy of the predicted eigenvalue sets in the complex plane. If the robust eigenvalues can be computed accurately, so can the robust flutter speed (although the quality of the robust flutter speed as such depends on the uncertainty description).

III. Uncertainty Modeling

The development of a suitable aerodynamic uncertainty description for the present configuration is described in more detail in [10,14]. The aerodynamic panels in the wing-tip region is divided into 20 different patches, as shown in Fig. 1b. Within each patch, the pressure difference coefficients are allowed to vary in a uniform manner according to $\Delta c_p = (1 + w_j \delta_j)\Delta c_{p0}$, where Δc_{p0} are the nominal pressure difference coefficients, and $w_j > 0$ is a real-valued uncertainty bound that scales the complex valued uncertainty parameter δ_j such that $|\delta_i| \leq 1$. Thus, a bound $w_j = 0.1$ means that the pressure coefficients in patch j are allowed to vary up to 10% in a uniform manner. This leads to an uncertain frequency-domain aerodynamic matrix in the form

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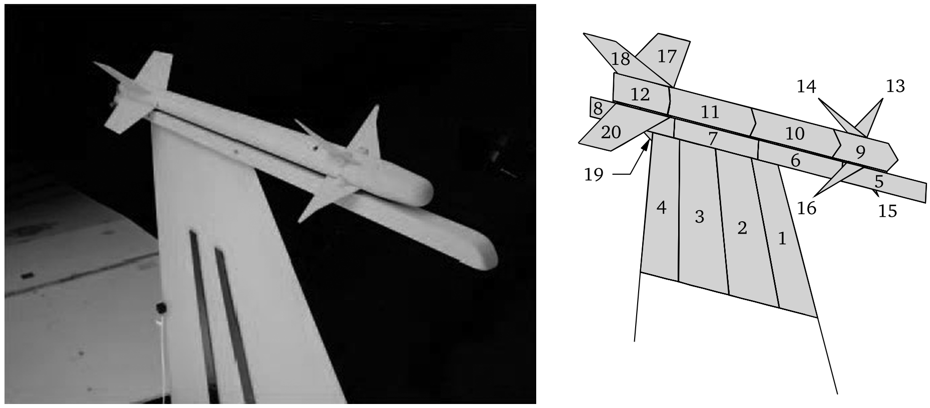

a) b)

Fig. 1 Photograph of wind-tunnel model and illustration of aerodynamic patches in the wing-tip region.

$$Q(ik,\delta) = Q_0(ik) + \sum_{j=1}^{n_p} w_j \delta_j Q_j(ik) \tag{1}$$

where δ is the vector of uncertainty parameters, n_p is the number of included patches, k is the reduced frequency, $Q_0(ik)$ is the nominal matrix, and $Q_j(ik)$ are aerodynamic perturbation matrices that determine the influence of the different patches.

By introducing uncertainty in the mass properties of the missile, a similar expression can be derived for the mass matrix. Uncertainty in three different parameters is considered: the mass m, the pitch moment of inertia J with respect to the center of mass, and the location x of the center of mass in the streamwise direction (with respect to a certain grid point in the model). Similar to the aerodynamic case, the parameters are allowed to vary according to

$$m = (1 + w_m \delta_m) m_0 \qquad J = (1 + w_J \delta_J) J_0 \qquad x = (1 + w_x \delta_x) x_0 \tag{2}$$

where m_0, J_0 ,and x_0 are the nominal parameters, and $w_i > 0$ are real-valued bounds such that the real-valued uncertainty parameters $\delta_i \in (-1, 1)$. The mass matrix depends linearly on the mass m, the static moment mx, and the mass moment of inertia $J + mx^2$ and can therefore be written in the form

$$\begin{aligned} M(\delta) = {} & M_0 + \hat{\delta}_J M_1 + \hat{\delta}_m M_2 + \hat{\delta}_x M_3 \\ & + \hat{\delta}_m \hat{\delta}_x M_4 + \hat{\delta}_x^2 M_5 + \hat{\delta}_m \hat{\delta}_x^2 M_6 \end{aligned} \tag{3}$$

where $\hat{\delta}_i = w_i \delta_i$, M_0 is the nominal mass matrix and M_1–M_6 is a set of perturbation matrices. Expression (3) was derived by evaluating the consistent mass matrix in NASTRAN for a set of different parameter values (m, J, and x) and solving a linear system of equations for the perturbation matrices M_1–M_6. The nominal modal basis was used for projection of the computed mass matrices to the modal subspace. The influence of mode-shape variations due to the varying mass properties is thus neglected, which is a good approximation as long as the variations are small [15].

IV. Robust Eigenvalue Analysis

Using the uncertain aerodynamic and mass matrices defined in Eqs. (1) and (3), respectively, leads to the Laplace-domain equations of motion:

$$F(p,\delta)\eta = [M(\delta)p^2 + (L/V)^2 K_0 - (\rho L^2/2) Q(ik,\delta)]\eta = 0 \tag{4}$$

where K_0 is the nominal stiffness matrix, L is the aerodynamic reference length, V is the airspeed, ρ is the air density, η is the vector of modal displacements, and $p = g + ik$ is the nondimensional Laplace variable with damping g and reduced frequency k. Equation (4) is an uncertain eigenvalue problem that depends on the unknown but bounded uncertainty parameters δ_k in the vector $\delta \in D$, where the set

$$D = \{\delta\colon |\delta_k| \le 1 \quad \text{and} \quad \delta_k \text{ real or complex}\} \tag{5}$$

The eigenvalues of the uncertain problem are defined as the values of p making the flutter matrix $F(p,\delta)$ singular for some $\delta \in D$. Consequently, p is an eigenvalue if $\det[F(p,\delta)] = 0$ for some $\delta \in D$. Clearly, if all eigenvalues of the uncertain problem have a negative real part, $g < 0$, the system is robustly stable subject to the uncertainty (no $\delta \in D$ can then destabilize the system).

For $\delta = 0$, the problem is reduced to the standard flutter problem without uncertainty, and each mode is represented by a single nominal eigenvalue. When uncertainty is introduced, each eigenvalue expands to a set of feasible eigenvalues in the complex plane [1]. Consequently, the nominal eigenvalue is always an interior point of the corresponding eigenvalue set. If the uncertainty parameters are few, it is (in principle) possible to compute the eigenvalue sets by performing a systematic sweep of the parameter space. This means that the eigenvalue problem (4) is solved for a discretized set of parameter values $\delta^j \in D$ in order to visualize the feasible solutions in the complex plane. In this work, the modified p-k solver described in [6] was used for this purpose. The p-k solutions can then be compared with the corresponding μ-p solutions.

The μ-p method exploits structured-singular-value analysis [2] to investigate if some uncertainty $\delta \in D$ can make the flutter determinant $\det[F(p,\delta)] = 0$ for a given value of p. If this is true, then p is an eigenvalue according to the definition above. As described in [1], the first step is to pose Eq. (4) in an equivalent feedback form defined by a system matrix $P(p)$ and a block-structured uncertainty matrix $\Delta(\delta)$. By computing the structured singular value μ of $P(p)$ subject to the block structure of $\Delta(\delta)$, it can be shown that p is an eigenvalue of the uncertain eigenvalue problem if

$$\mu(p) = \mu[P(p)] \ge 1 \tag{6}$$

Consequently, by evaluating the real-valued scalar function $\mu(p)$ for a given value of p, it is possible to determine if p is an eigenvalue or not. This result can be exploited, for example, to compute the boundary of an eigenvalue set in the complex plane. First, the nominal eigenvalue is computed using a standard method. By searching in different directions in the complex plane, it is then possible to compute the points on the boundary where $\mu(p) = 1$ and visualize the result. This approach is used in the present study.

Note that the approach based on p-k analysis described above quickly becomes infeasible when the number of uncertain parameters increase, while the approach based on μ-p analysis does not (although a larger number of parameters makes it more expensive to compute μ). In practice, the criterion (6) must be evaluated using a computable upper bound for μ [11]. This means that the predicted eigenvalue sets may expand somewhat in the complex plane, leading to a certain amount of conservatism in the robust analysis. The quality of the upper bound depends on the uncertainty description: in particular, if it is purely complex or mixed real/complex. For this reason, the main objective of the present study is to investigate what impact a mixed structural/aerodynamic (real/complex) uncertainty description may have on the quality of the computed μ-p eigenvalue sets.

V. Results

In this work, the required system matrix $P(p)$ and corresponding block structure of the uncertainty matrix Δ was derived in a similar way as described in [16]. In particular, a singular-value decomposition of the perturbation matrices M_1–M_6 in Eq. (3) was performed in order to obtain a minimum-dimension uncertainty matrix Δ (reducing the computational cost). Note that the higher-order terms in Eq. (3) lead to repeated uncertainty blocks in Δ, which increases the computational cost compared with a case with only linear terms. The μ solver `mussv` in MATLAB was used for the μ-p

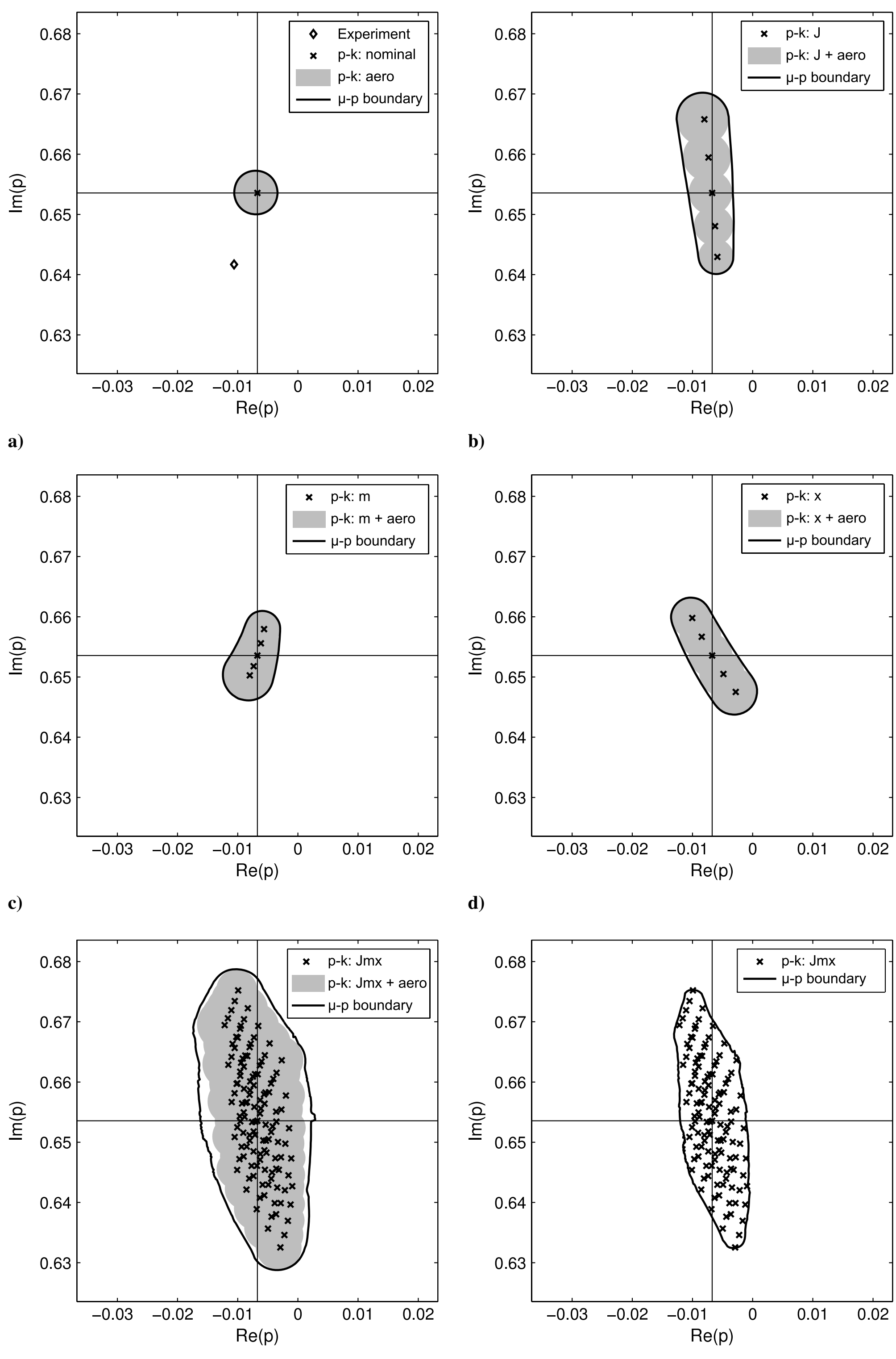

Fig. 2 Eigenvalue sets for the second mode at $V = 25$ m/s due to various model uncertainties.

analysis, using the default settings. The MATLAB μ solver is very efficient for medium-sized problems (less than 100 perturbations); on the other hand, it does not allow for much tuning to enhance the quality of the computed bounds.

A. Aerodynamic Uncertainty

To allow for a parameter sweep with p-k analysis, the aerodynamic uncertainty was first limited to patch number 1 in Fig. 1b. The eigenvalue problem (4) will then depend on one complex uncertainty parameter δ_Q. First, a parameter sweep taking only aerodynamic uncertainty into account was performed by solving the eigenvalue problem for a set of parameter values $\delta_Q = e^{i\varphi}$ on the complex unit circle, where $\varphi \in (0, 2\pi)$. As shown in Fig. 2a, the corresponding eigenvalue set for the second mode at the airspeed $V = 25$ m/s is almost a perfect circle (the shaded region). Note that the nominal eigenvalue, located in the center of the graph, is an interior point of the eigenvalue set.

Here, a so-called g validation [1] of the aerodynamic uncertainty description was performed, which basically means that the uncertainty bound is increased until some eigenvalue satisfies $g = g_{\text{exp}}$, where g_{exp} is the real part of a corresponding experimental eigenvalue p_{exp}. In the present case, an eigenvalue obtained in wind-tunnel testing at $V = 25$ m/s was used for the model validation (marked in Fig. 2a). Using patch number 1 only, a bound $w_Q = 0.49$ was required to validate the model, which is the same result as reported in [10]. The result of the model validation can be confirmed in Fig. 2a, where the shaded region exactly touches a thought line $g = g_{\text{exp}}$ in the complex plane.

Next, μ-p analysis was used to compute the boundary of the same eigenvalue set, represented by the solid boundary in Fig. 2a. A perfect match between p-k and μ-p analysis is obtained. This result is expected in this simple case, since μ is equal to the spectral radius ρ in the case of one complex perturbation [2]. Still, this example illustrates the influence of the aerodynamic uncertainty, as well as the principle of g validation, and also forms the basis for the subsequent p-k parameter sweeps.

B. Restricted Mass Uncertainty

The uncertainty bounds for the mass uncertainty were, more or less arbitrarily, set to $w_m = w_J = 0.1$ and $w_x = 0.02$. This means that a 10% uncertainty was assumed for the missile mass m and mass moment of inertia J, respectively, and that a 2% uncertainty was assumed for the location of the center of mass x. The different perturbations were introduced gradually in the analysis. First, a linear mass perturbation was taken into account by combining the aerodynamic uncertainty with uncertainty in J only. A p-k parameter sweep was performed by combining the mass perturbations $\delta_J = (-1, -0.5, 0, 0.5, 1)$ with a larger number of values of δ_Q on the complex unit circle (as described above). As seen in Fig. 2b, five circular regions (one for each value of δ_J) appear in the complex plane. Although a larger number of values of δ_J could be used, the shape of the exact eigenvalue set is apparent. In this case, the perturbation in J mainly affects the frequency of the mode, and the observed variation in damping is mainly caused by the aerodynamic uncertainty.

Again, the computed μ-p boundary show an excellent agreement with the p-k solution (shown as the solid boundary in Fig. 2b). Apparently, the μ solver computes a very tight upper bound when one real parameter is taken into account. This was also true for perturbations in m and x, respectively, as displayed in Figs. 2c and 2d. Note that a quadratic term in δ_x is present in the mass matrix in Eq. (3), which increases the dimension of the corresponding μ problem somewhat. Still, the agreement in Fig. 2d is remarkable. Also note that the system is not robustly stable at $V = 25$ m/s, subject to the uncertainty in x (and the aerodynamics), since some eigenvalues in the set have a positive real part.

C. Full Mass Uncertainty

Next, uncertainties in all mass parameters were taken into account simultaneously, according to the mass matrix in Eq. (3). The parameter values $\delta_J = \delta_m = \delta_x = (-1, -0.5, 0, 0.5, 1)$ were combined with a set of values of δ_Q on the complex unit circle to perform a sweep of the parameter space. About 10^4 p-k eigenvalues were computed to obtain the eigenvalue set in Fig. 2e. Although it is more crude than the previous ones, it is still possible to discern the shape of the exact eigenvalue set.

In this case, where one complex and three real perturbations are taken into account, the quality of the μ upper bound is somewhat deteriorated (see Fig. 2e). While the upper bound as such is still fairly tight, which is necessary to obtain useful bounds on the aeroelastic damping, the computed boundary is not smooth. As a result, optimization-based algorithms for computing the eigenvalues with minimum or maximum damping, like the one described in [1], will most likely fail. Consequently, if efficient and reliable robust analysis should be performed, either a dedicated μ solver (that guarantees a smooth boundary) or some approximate method to extract the damping bounds must be developed. Nevertheless, the obtained results are very encouraging, since the μ-p eigenvalue sets closely bound the exact p-k eigenvalue sets.

D. Further Investigations

In addition to the cases accounted for so far, the capability of μ-p analysis to compute the eigenvalue set for a case dominated by mass uncertainty was also investigated. The aerodynamic uncertainty bound was reduced to $w_Q = 0.01$ (1% uncertainty in the first patch) and the resulting eigenvalue set was computed. In this case, the

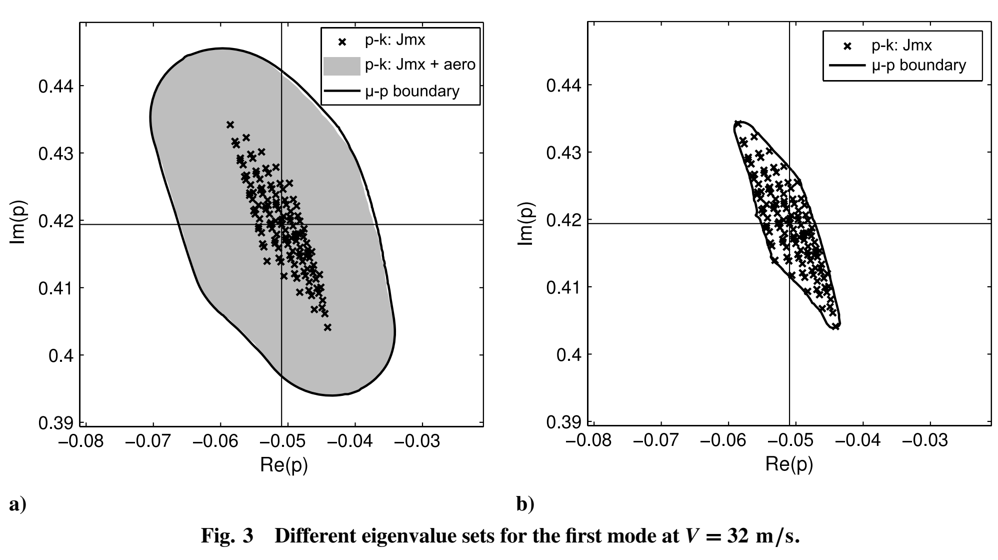

Fig. 3 Different eigenvalue sets for the first mode at $V = 32$ m/s.

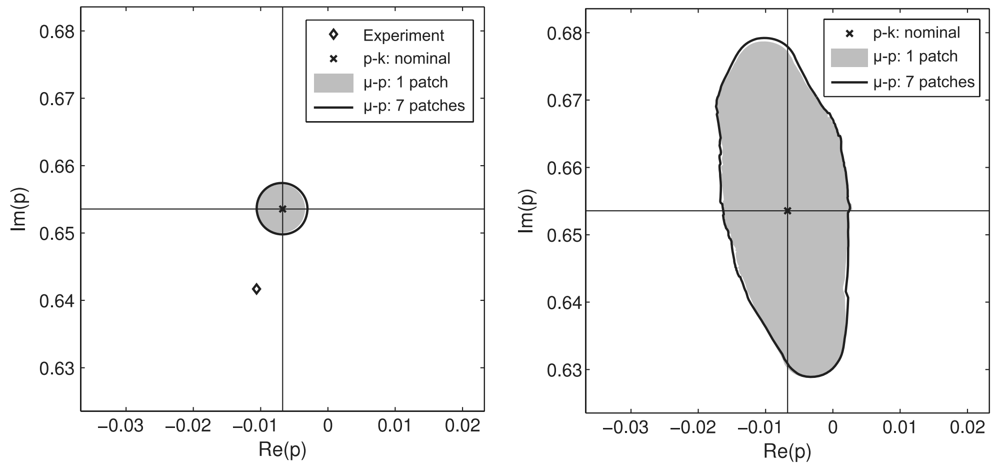

Fig. 4 Plots of a) aerodynamic model validation at $V = 25$ m/s and b) resulting eigenvalue sets when mass uncertainty is included.

shaded circular regions from the *p-k* analysis are so small that they are not visible in Fig. 2f. However, the μ-p boundary is once again found to be a good approximation of the exact eigenvalue set. This indicates that is possible to treat cases with only mass/stiffness uncertainty by introducing a small complex perturbation. This is not difficult to justify in practice, because some aerodynamic uncertainty is always present.

Some results for the first mode are also included to support the usefulness of μ-p analysis. In Fig. 3a, the eigenvalue sets for the first mode at $V = 32$ m/s are displayed (using the same uncertainty bounds as for the second mode). This mode is clearly more sensitive to the aerodynamic uncertainty, which can be the reason for the smoother boundary of the μ-p eigenvalue set. Again, the agreement between the *p-k* and μ-p solutions is remarkable, including the case with 1% aerodynamic uncertainty in Fig. 3b.

Finally, a case with a larger number of aerodynamic patches was treated. As described in [10], a more realistic aerodynamic uncertainty description is obtained by including the seven patches (1, 3, 4, 9, 12, 17, and 18) in Fig. 1b. These patches were selected because they were found to have the largest impact on the damping of the second (critical) mode. Taking this aerodynamic uncertainty into account, a model with seven complex and three real perturbations is obtained. Here, a *p-k* parameter sweep is virtually impossible, since about 10^{15} eigenvalues have to be computed to obtain the same resolution as in the previous sweeps. However, μ-p analysis can still be applied to compute the eigenvalue sets.

Assuming the same uncertainty bound for all seven patches, a g validation of the aerodynamic uncertainty model was performed as described in [1]. A value of $w_Q = 0.18$ was required to validate the experimental damping, which was also found in [10]. As seen in Fig. 4a, the result is another circular eigenvalue set that can hardly be distinguished from the one based on one single patch. Taking the full mass uncertainty into account, the eigenvalue sets based on one patch and seven patches, respectively, differ only slightly (see Fig. 4b). This means that as long as model validation is performed, the eigenvalue set appears to be quite insensitive to details in the aerodynamic uncertainty description. This conclusion was also drawn with respect to the robust flutter boundaries in a previous study [10].

As a final remark, a μ value for the case with seven aerodynamic patches and full mass uncertainty was typically computed in less than 2 s on a modern personal computer. Thus, despite the larger number of patches, the computational cost using μ-p analysis was still very modest.

VI. Conclusions

This work has shown that μ-p analysis can be used to compute accurate eigenvalue sets in the complex plane in the presence of mixed aerodynamic/structural uncertainty in the flutter problem. Comparison between μ-p solutions and exact *p-k* solutions showed an excellent agreement, even in cases dominated by structural (real-valued) uncertainty. However, it was also observed that the numerical quality of the μ upper bound may be a concern in the development of tools for reliable and efficient robust flutter analysis. Nevertheless, the obtained results are very promising and confirm the efficiency and practical usefulness of μ-p analysis.

Acknowledgments

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References

[1] Borglund, D., "Robust Eigenvalue Analysis Using the Structured Singular Value: The μ-p Flutter Method," *AIAA Journal*, Vol. 46, No. 11, 2008, pp. 2806–2813.
doi:10.2514/1.35859
[2] Zhou, K., Doyle, J. C., and Glover, K., *Robust and Optimal Control*, Prentice–Hall, Upper Saddle River, NJ, 1996, Chap. 11.
[3] Lind, R., "Match-Point Solutions for Robust Flutter Analysis," *Journal of Aircraft*, Vol. 39, No. 1, 2002, pp. 91–99.
doi:10.2514/2.2900
[4] Borglund, D., "The μ-k Method for Robust Flutter Solutions," *Journal of Aircraft*, Vol. 41, No. 5, 2004, pp. 1209–1216.
doi:10.2514/1.3062
[5] Borglund, D., "Upper-Bound Flutter Speed Estimation Using the μ-k Method," *Journal of Aircraft*, Vol. 42, No. 2, 2005, pp. 555–557.
doi:10.2514/1.7586
[6] Bäck, P., and Ringertz, U., "Convergence of Methods for Nonlinear Eigenvalue Problems," *AIAA Journal*, Vol. 35, No. 6, 1997, pp. 1084–1087.
doi:10.2514/2.200
[7] Chen, P. C., "Damping Perturbation Method for Flutter Solution: The g-Method," *AIAA Journal*, Vol. 38, No. 9, 2000, pp. 1519–1524.
doi:10.2514/2.1171
[8] Carlsson Leijonhufvud, M., and Karlsson, A., "Robust Aeroelastic Analysis of the Gripen Fighter Including Flight Test Model Validation," CEAS/AIAA/KTH International Forum on Aeroelasticity and Structural Dynamics 2007, Paper IF-019, Stockholm, June 2007.
[9] Carlsson Leijonhufvud, M., and Karlsson, A., "Industrial Application of Robust Aeroelastic Analysis," CEAS/AIAA/USAF International Forum on Aeroelasticity and Structural Dynamics 2009, Paper IFASD-2009-47, Seattle, WA, June 2009.
[10] Heinze, S., Ringertz, U., and Borglund, D., "Assessment of Uncertain External Store Aerodynamics Using μ-p Flutter Analysis," *Journal of*

Aircraft, Vol. 46, No. 3, 2009, pp. 1062–1067.
doi:10.2514/1.39158

[11] Young, P., Newlin, M., and Doyle, J., "Practical Computation of the Mixed μ Problem," *Proceedings of the American Control Conference*, Vol. 3, Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1992, pp. 2190–2194.

[12] NX NASTRAN, Software Package, Ver. 6.1, Siemens PLM, Cleveland, OH, 2009.

[13] MATLAB, Software Package, Ver. 7.3, The MathWorks, Inc., Natick, MA, 2006.

[14] Borglund, D., and Ringertz, U., "Efficient Computation of Robust Flutter Boundaries Using the μ-k Method," *Journal of Aircraft*, Vol. 43, No. 6, 2006, pp. 1763–1769.
doi:10.2514/1.20190

[15] Heinze, S., and Borglund, D., "Robust Flutter Analysis Considering Mode Shape Variations," *Journal of Aircraft*, Vol. 45, No. 3, 2008, pp. 1070–1074.
doi:10.2514/1.28728

[16] Heinze, S., "Assessment of Critical Fuel Configurations Using Robust Flutter Analysis," *Journal of Aircraft*, Vol. 44, No. 6, 2007, pp. 2034–2039.
doi:10.2514/1.30500